A.2 Stetige Verteilungen

A.2.1 Uniforme Verteilung (Gleichverteilung)

Bezeichnung: $X \sim U_{a,b}$, $U(a,b)$

Parameter: $a, b \in \mathbb{R}$, $a < b$

Merkmalraum: $M_X = (a,b)$, $M_X = [a,b]$

Dichte:

$$f(x) = \frac{1}{b-a} I_{(a,b)}(x)$$

Verteilungsfunktion:

$$F(x) = \frac{x-a}{b-a} I_{(a,b)}(x) + I_{[b,\infty)}(x)$$

Mittelwert:

$$\mathbb{E}(X) = \frac{a+b}{2}$$

Varianz:

$$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

Zentrale Momente:

$$\mathbb{E}\big[X - \mathbb{E}(X)\big]^k = \begin{cases} 0 & k = 1, 3, \ldots \\ \dfrac{(b-a)^k}{2^k(k+1)} & k = 2, 4, \ldots \end{cases}$$

Momentenerzeugende Funktion:

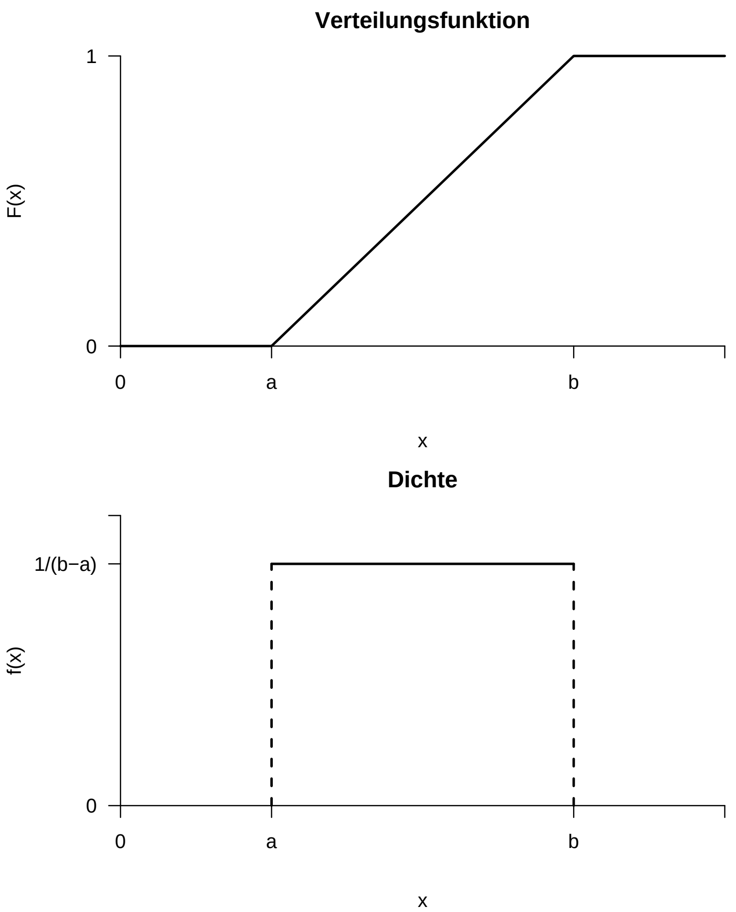
$$m(t) = \frac{e^{bt} - e^{at}}{(b-a)t}, \quad t \in \mathbb{R} \quad [m(0) = 1]$$

Quantile:

$$x_p = a + (b-a)p, \quad 0 < p < 1$$

R–Funktionen: `min` $\widehat{=}$ a, `max` $\widehat{=}$ b

```
dunif(x, min=0, max=1, log = FALSE)
punif(q, min=0, max=1, lower.tail = TRUE, log.p = FALSE)
qunif(p, min=0, max=1, lower.tail = TRUE, log.p = FALSE)
runif(n, min=0, max=1)
```


Verteilungsfunktion
1
0
F(x)
0
a
b
x
Dichte
1/(b−a)
0
f(x)
0
a
b
x

A.2.2 Exponentialverteilung

Bezeichnung: $X \sim Ex_\tau$, $Ex(\tau)$, Ex_λ, $Ex(\lambda)$

Parameter: $\tau > 0$ (Skalierungsparameter), $\lambda := 1/\tau$ (Ausfallrate)

Merkmalraum: $M_X = (0, \infty)$, $M_X = [0, \infty)$

Dichte:

$$f(x) = \frac{1}{\tau} e^{-x/\tau} I_{(0,\infty)}(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$$

Verteilungsfunktion:

$$F(x) = \bigl(1 - e^{-x/\tau}\bigr) I_{(0,\infty)}(x) = \bigl(1 - e^{-\lambda x}\bigr) I_{(0,\infty)}(x)$$

Mittelwert:

$$\mathbb{E}(X) = \tau = \frac{1}{\lambda}$$

Varianz:

$$\mathrm{Var}(X) = \tau^2 = \frac{1}{\lambda^2}$$

Momente:

$$\mathbb{E}\bigl(X^k\bigr) = \tau^k k! = \frac{k!}{\lambda^k}, \quad k = 1, 2, \ldots$$

Momentenerzeugende Funktion:

$$m(t) = \frac{1}{1 - \tau t} = \frac{\lambda}{\lambda - t}, \quad t < \lambda$$

Quantile:

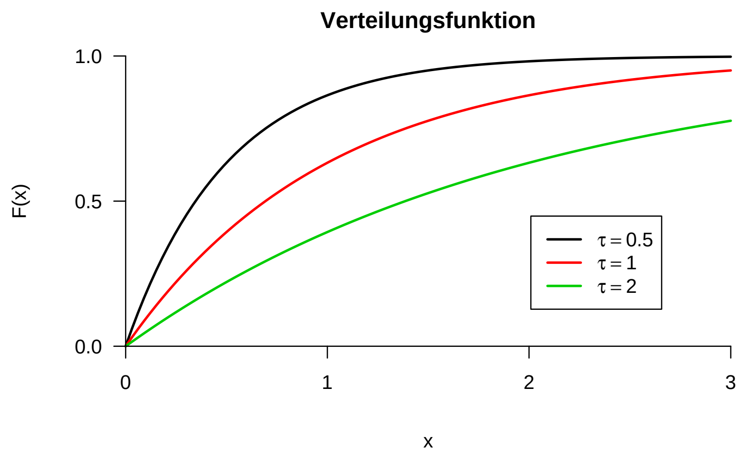
$$x_p = -\tau \ln(1 - p) = -\frac{\ln(1 - p)}{\lambda}, \quad 0 < p < 1$$

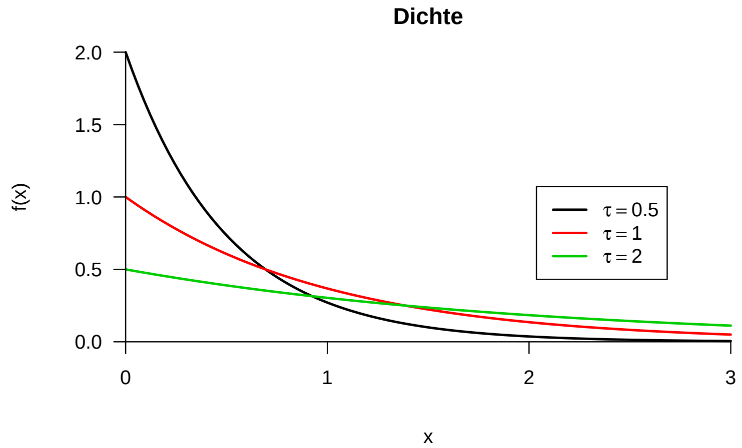
Additionsthorem: $X_i \sim Ex_\tau$, $i = 1, 2, \ldots, n$, ua.:

$$\sum_{i=1}^{n} X_i \sim Er_{n,\tau} \quad \text{(Erlangverteilung)}$$

R–Funktionen: `rate` $\widehat{=}\ \lambda$

```
dexp(x, rate = 1, log = FALSE)
pexp(q, rate = 1, lower.tail = TRUE, log.p = FALSE)
qexp(p, rate = 1, lower.tail = TRUE, log.p = FALSE)
rexp(n, rate = 1)
```


Verteilungsfunktion
1.0
0.5
0.0
F(x)
0
1
2
3
x
τ = 0.5
τ = 1
τ = 2


Dichte
2.0
1.5
1.0
0.5
0.0
f(x)
0
1
2
3
x
τ = 0.5
τ = 1
τ = 2

A.2.3 Normalverteilung (Gaußverteilung)

Bezeichnung: $X \sim N(\mu, \sigma^2)$

Parameter: $\mu \in \mathbb{R}$ (Lageparameter), $\sigma > 0$ (Skalierungsparameter)

Merkmalraum: $M_X = \mathbb{R}$

Dichte:

$$N(0,1): \quad \varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \quad x \in \mathbb{R}$$

$$N(\mu,\sigma^2): \quad f(x) = \frac{1}{\sigma}\varphi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2}, \quad x \in \mathbb{R}$$

Verteilungsfunktion:

$$N(0,1): \quad \Phi(x) = \int\limits_{-\infty}^{x} \varphi(u)\, du = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-u^2/2}\, du, \quad x \in \mathbb{R}$$

$$N(\mu,\sigma^2): \quad F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right), \quad x \in \mathbb{R}$$

Mittelwert:

$$\mathbb{E}(X) = \mu$$

Varianz:

$$\mathrm{Var}(X) = \sigma^2$$

Zentrale Momente:

$$\mathbb{E}\big[X - \mathbb{E}(X)\big]^k = \begin{cases} 0 & k = 1, 3, \ldots \\ \dfrac{k!}{(k/2)!}\,\dfrac{\sigma^k}{2^{k/2}} & k = 2, 4, \ldots \end{cases}$$

Momentenerzeugende Funktion:

$$m(t) = e^{\mu t + \sigma^2 t^2/2}, \quad t \in \mathbb{R}$$

Quantile:

$$N(0,1): \quad u_p \quad \big[\text{häufig auch } z_p\big]$$

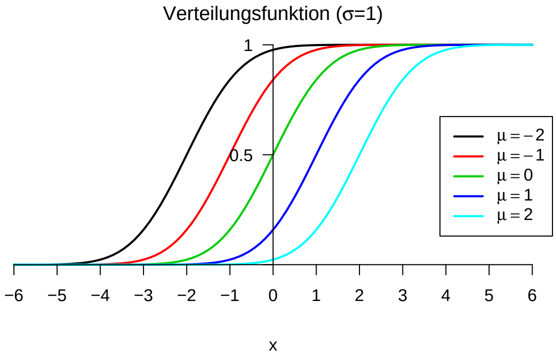
$$N(\mu,\sigma^2): \quad x_p = \mu + \sigma u_p$$

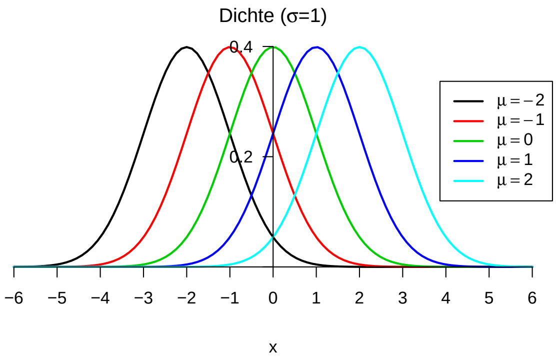
Additionstheorem: $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1, 2, \ldots, n$, ua.; $c_i \in \mathbb{R}$:

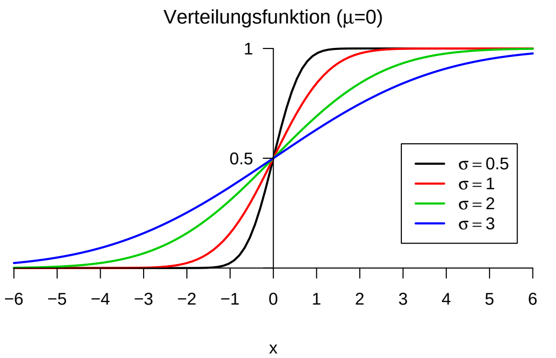
$$\sum_{i=1}^{n} c_i X_i \sim N\left(\sum_{i=1}^{n} c_i \mu_i, \sum_{i=1}^{n} c_i^2 \sigma_i^2\right)$$

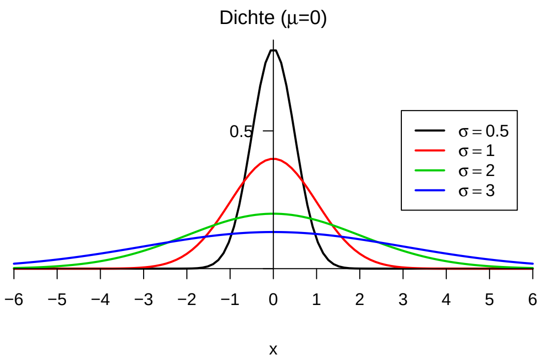
R–Funktionen: `mean` $\widehat{=}\ \mu$, `sd` $\widehat{=}\ \sigma$

```
dnorm(x, mean=0, sd=1, log = FALSE)
pnorm(q, mean=0, sd=1, lower.tail = TRUE, log.p = FALSE)
qnorm(p, mean=0, sd=1, lower.tail = TRUE, log.p = FALSE)
rnorm(n, mean=0, sd=1)
```

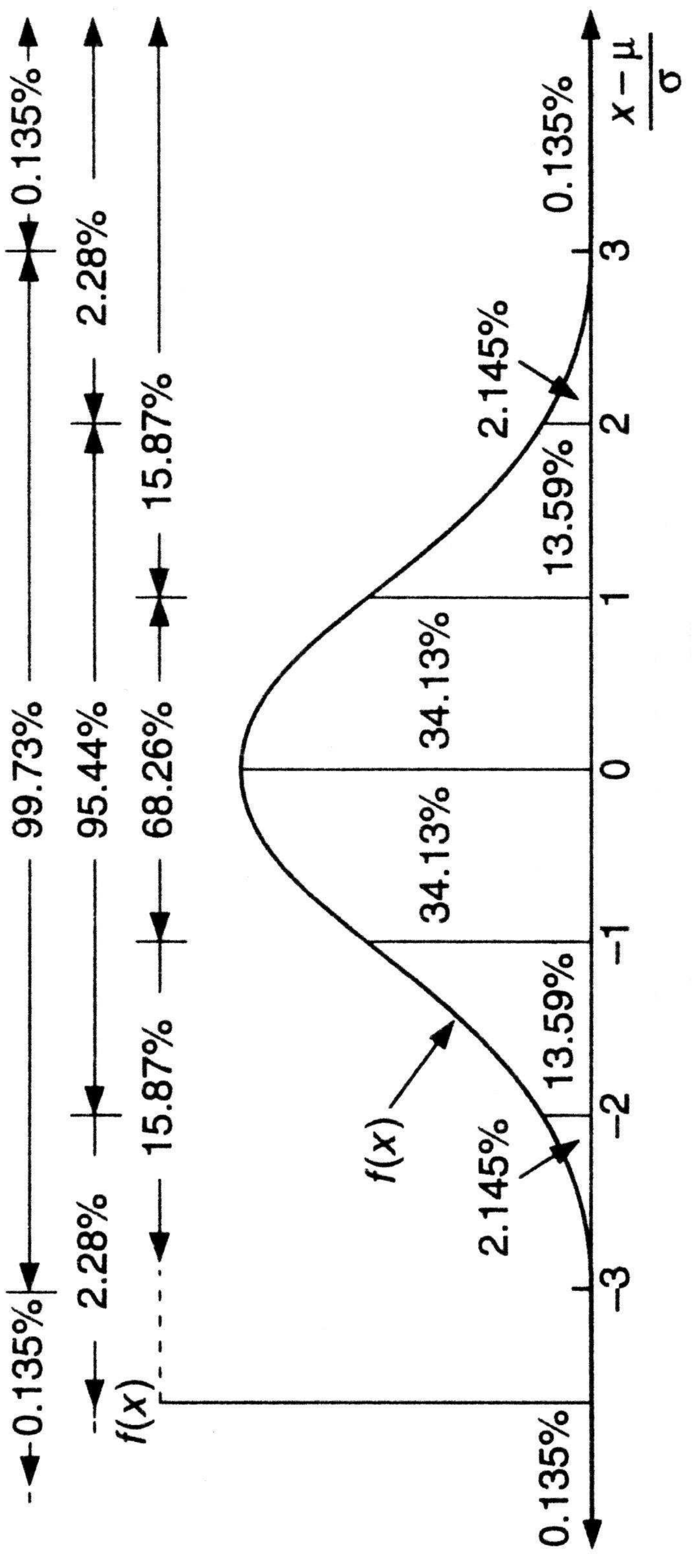








Zusatz: Die folgende Übersicht (aus D. Bissell: *Statistical Methods for SPC and TQM*, 1994) zeigt eine Reihe von häufig benötigten Wahrscheinlichkeiten im Zusammenhang mit der Normalverteilung.



A.2.4 Logarithmische Normalverteilung (Log–Normalverteilung)

Bezeichnung: $X \sim LN(\mu, \sigma^2)$, $L(\mu, \sigma^2)$

Parameter: $\mu \in \mathbb{R}$ (e^μ Skalierungsparameter), $\sigma > 0$ (Formparameter)

Merkmalraum: $M_X = \mathbb{R}^+$

Dichte:

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}}\, e^{-(\ln x - \mu)^2/2\sigma^2}, \quad x > 0$$

Verteilungsfunktion:

$$F(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right), \quad x > 0$$

Mittelwert:

$$\mathbb{E}(X) = e^{\mu + \sigma^2/2}$$

Varianz:

$$\mathrm{Var}(X) = e^{2\mu+\sigma^2}\left[e^{\sigma^2} - 1\right]$$

Momente:

$$\mathbb{E}(X^k) = e^{k\mu + k^2\sigma^2/2}, \quad k = 1, 2, \ldots$$

Quantile:

$$x_p = e^{\mu + \sigma u_p}, \quad 0 < p < 1$$

Modus:

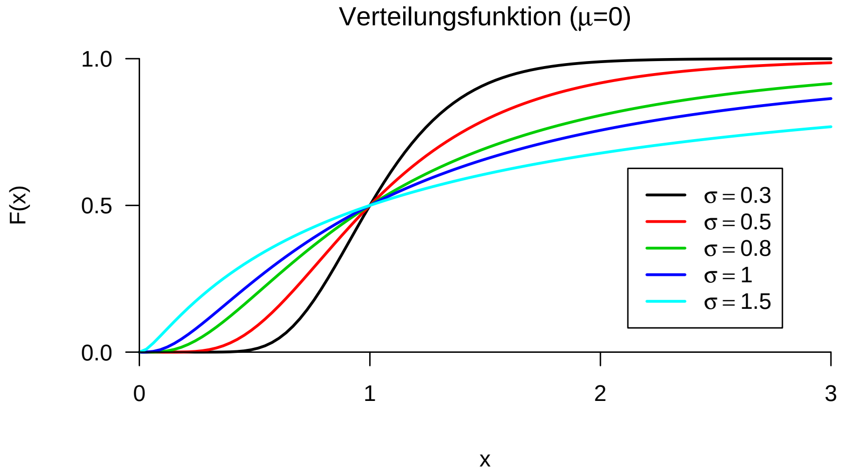
$$x_{\mathrm{mod}} = e^{\mu - \sigma^2}$$

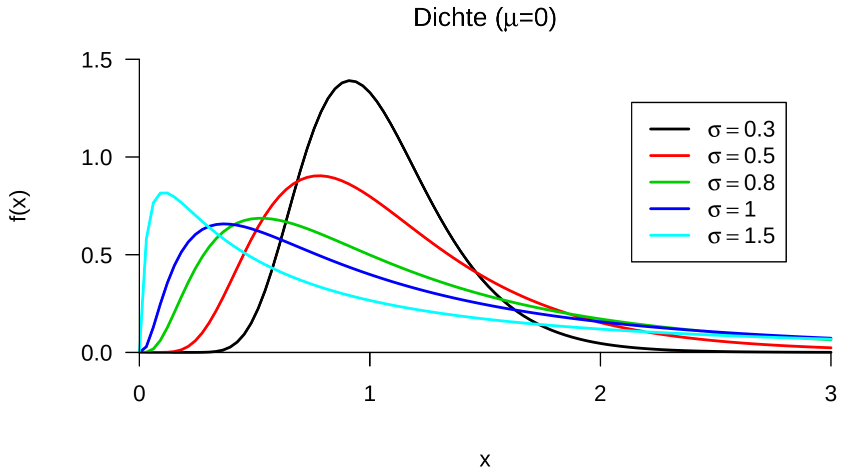
Produkttheorem: $X_i \sim LN(\mu_i, \sigma_i^2)$, $i = 1, 2, \ldots, n$, ua.:

$$\prod_{i=1}^{n} X_i \sim LN\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$$

R–Funktionen: `meanlog` $\widehat{=}\ \mu$, `sdlog` $\widehat{=}\ \sigma$

```
dlnorm(x, meanlog = 0, sdlog = 1, log = FALSE)
plnorm(q, meanlog = 0, sdlog = 1, lower.tail = TRUE, log.p = FALSE)
qlnorm(p, meanlog = 0, sdlog = 1, lower.tail = TRUE, log.p = FALSE)
rlnorm(n, meanlog = 0, sdlog = 1)
```


Verteilungsfunktion (μ=0)
F(x)
x
σ = 0.3
σ = 0.5
σ = 0.8
σ = 1
σ = 1.5


Dichte (μ=0)
f(x)
x
σ = 0.3
σ = 0.5
σ = 0.8
σ = 1
σ = 1.5

A.2.5 Gammaverteilung

Bezeichnung: $X \sim \gamma(\alpha, \beta)$, $Gam(\alpha, \beta)$

Parameter: $\alpha > 0$ (Formparameter), $\beta > 0$ (Skalierungsparameter)

Merkmalraum: $M_X = \mathbb{R}^+$

Gammafunktion:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt, \quad x > 0$$

$$\Gamma(x+1) = x\Gamma(x), \quad n \in \mathbb{N} : \Gamma(n) = (n-1)!, \quad \Gamma(1/2) = \sqrt{\pi}$$

Dichte:

$$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\, \beta^\alpha}, \quad x > 0$$

Verteilungsfunktion:

$$F(x) = \frac{1}{\Gamma(\alpha)\, \beta^\alpha} \int_0^x t^{\alpha-1} e^{-t/\beta}\, dt, \quad x > 0 \qquad \text{[unvollständige Gammafunktion]}$$

Mittelwert:

$$\mathbb{E}(X) = \alpha\beta$$

Varianz:

$$\mathrm{Var}(X) = \alpha\beta^2$$

Momente:

$$\mathbb{E}(X^k) = \frac{\beta^k \Gamma(\alpha+k)}{\Gamma(\alpha)}, \quad k = 1, 2, \ldots$$

Momentenerzeugende Funktion:

$$m(t) = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < \frac{1}{\beta}$$

Modus:

$$x_{\mathrm{mod}} = (\alpha - 1)\beta, \quad \alpha \geq 1$$

Additionstheorem: $X_i \sim \gamma(\alpha_i, \beta)$, $i = 1, 2, \ldots, n$, ua.:

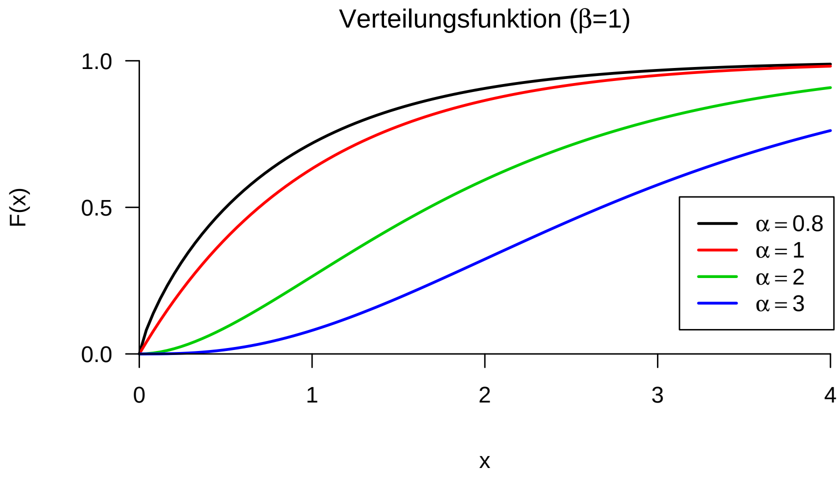
$$\sum_{i=1}^{n} X_i \sim \gamma\left(\sum_{i=1}^{n} \alpha_i, \beta\right)$$

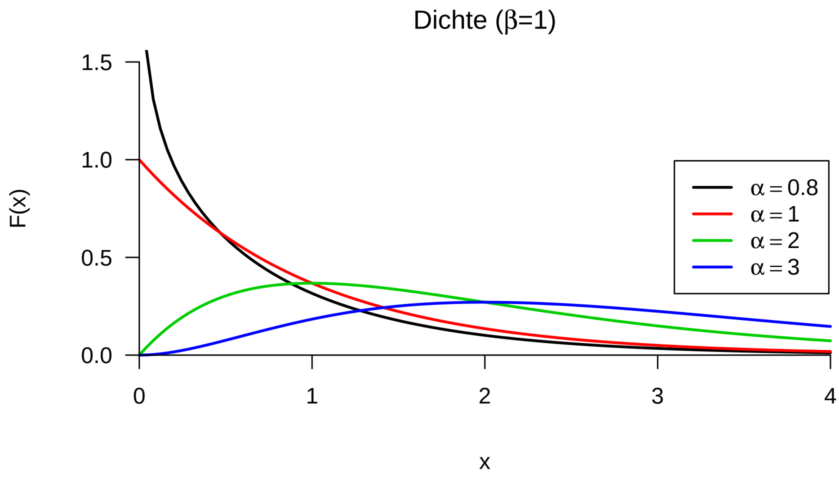
Spezialfälle:

$$\begin{aligned} \alpha = 1: &\ \gamma(1, \beta) \equiv Ex_\beta \text{ (Exponential)} \\ \alpha = r \in \mathbb{N}: &\ \gamma(r, \beta) \equiv Er_{r,\beta} \text{ (Erlang)} \\ \alpha = n/2\ (n \in \mathbb{N}),\ \beta = 2: &\ \gamma(n/2, 2) \equiv \chi^2_n \text{ (Chiquadrat)} \end{aligned}$$

R–Funktionen: `shape` $\widehat{=}\ \alpha$, `rate` $\widehat{=}\ 1/\beta$, `scale` $\widehat{=}\ \beta$

```
gamma(x)  # Gammafunktion
dgamma(x, shape, rate = 1, scale = 1/rate, log = FALSE)
pgamma(q, shape, rate = 1, scale = 1/rate, lower.tail = TRUE,log.p = FALSE)
qgamma(p, shape, rate = 1, scale = 1/rate, lower.tail = TRUE,log.p = FALSE)
rgamma(n, shape, rate = 1, scale = 1/rate)
```





A.2.6 Chiquadratverteilung (χ^2–Verteilung)

Bezeichnung: $X \sim \chi^2_n$, $\chi^2(n)$

Parameter: $n \in \mathbb{N}$ (Freiheitsgrade, Formparameter)

Merkmalraum: $M_X = \mathbb{R}^+$

Dichte:

$$f(x) = \frac{x^{n/2-1}e^{-x/2}}{\Gamma(n/2)\,2^{n/2}}, \quad x > 0$$

Mittelwert:

$$\mathbb{E}(X) = n$$

Varianz:

$$\mathrm{Var}(X) = 2n$$

Momente:

$$\mathbb{E}(X^k) = \frac{2^k\Gamma(n/2+k)}{\Gamma(n/2)}, \quad k = 1, 2, \ldots$$

Momentenerzeugende Funktion:

$$m(t) = \left(\frac{1}{1-2t}\right)^{n/2}, \quad t < \frac{1}{2}$$

Modus:

$$x_{\mathrm{mod}} = n - 2, \quad n \geq 2$$

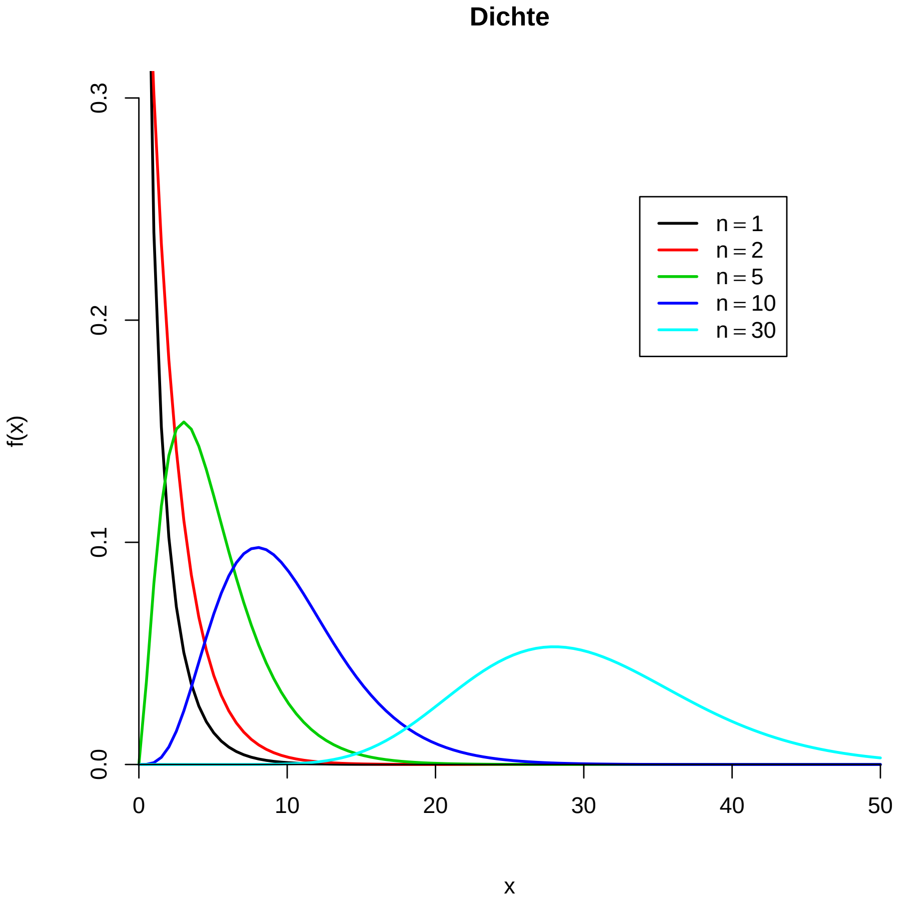
Spezialfall: $\chi^2_2 \equiv Ex_2$

Additionstheorem: $X_i \sim \chi^2_{n_i}$, $i = 1, 2, \ldots, K$, ua.:

$$\sum_{i=1}^{K} X_i \sim \chi^2_n, \; n = \sum_{i=1}^{K} n_i$$

R–Funktionen: `df` $\widehat{=}\ n$

```
dchisq(x, df, ncp=0, log = FALSE)
pchisq(q, df, ncp=0, lower.tail = TRUE, log.p = FALSE)
qchisq(p, df, ncp=0, lower.tail = TRUE, log.p = FALSE)
rchisq(n, df, ncp=0)
```


Dichte
n = 1
n = 2
n = 5
n = 10
n = 30
f(x)
x
0.0
0.1
0.2
0.3
0
10
20
30
40
50

A.2.7 t–Verteilung (Studentverteilung)

Bezeichnung: $X \sim t_n$, $t(n)$

Parameter: $n \in \mathbb{N}$ (Freiheitsgrade, Formparameter)

Merkmalraum: $M_X = \mathbb{R}$

Dichte:

$$f(x) = \frac{\Gamma\big((n+1)/2\big)}{\sqrt{n\pi}\,\Gamma(n/2)(1+x^2/n)^{(n+1)/2}}, \quad x \in \mathbb{R}$$

Mittelwert:

$$\mathbb{E}(X) = 0, \quad n > 1$$

Varianz:

$$\mathrm{Var}(X) = \frac{n}{n-2}, \quad n > 2$$

(Zentrale) Momente:

$$\mathbb{E}(X^k) = \begin{cases} 0 & n > k,\ k \text{ ungerade} \\ \dfrac{n^{k/2} B\big((k+1)/2, (n-k)/2\big)}{B(1/2, n/2)} & n > k,\ k \text{ gerade} \end{cases}$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \text{(Betafunktion)}$$

Quantile:

$$t_{n;\,p} = -t_{n;\,1-p}, \quad 0 < p < 1$$

Spezialfall: $t_1 \equiv C(0,1)$ (Cauchy–Verteilung)

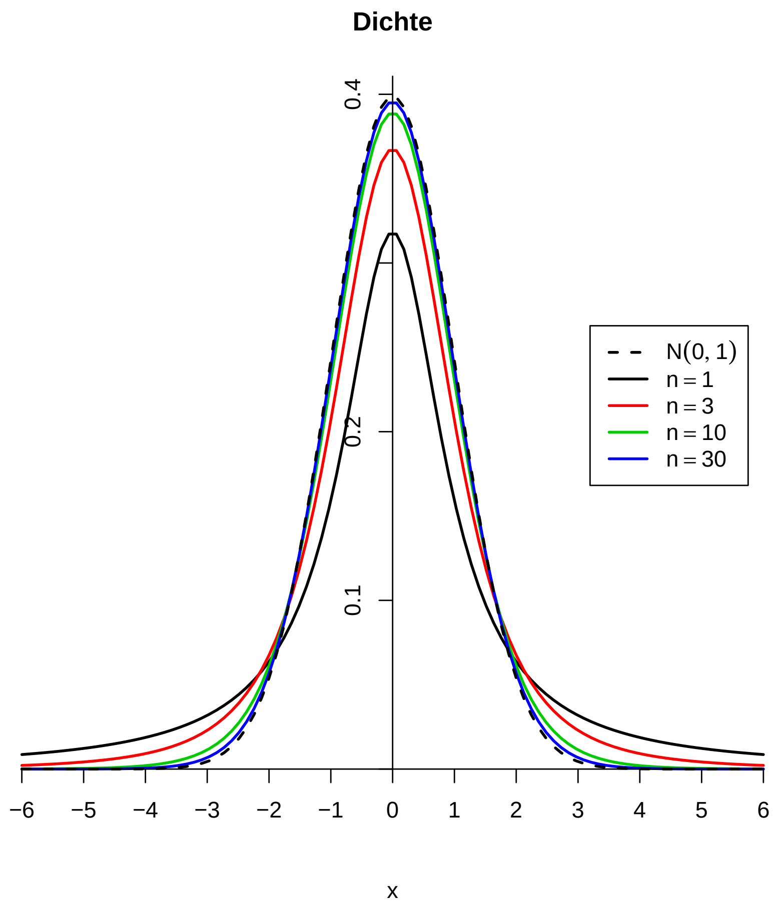
Beziehung zur Normalverteilung: $f(x|n)$ = Dichte der t_n–Verteilung

$$\lim_{n\to\infty} f(x|n) = \varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$

$$\lim_{n\to\infty} t_{n;\,p} = u_p \quad (0 < p < 1)$$

R–Funktionen: `df` $\widehat{=}\ n$

```
beta(a, b)  # Betafunktion
dt(x, df, ncp = 0, log = FALSE)
pt(q, df, ncp = 0, lower.tail = TRUE, log.p = FALSE)
qt(p, df, ncp = 0, lower.tail = TRUE, log.p = FALSE)
rt(n, df, ncp = 0)
```


Dichte
0.4
0.2
0.1
N(0, 1)
n = 1
n = 3
n = 10
n = 30
−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6
x

A.2.8 F–Verteilung (Fisher–Verteilung)

Bezeichnung: $X \sim F_{m,n}$, $F(m,n)$

Parameter: $m, n \in \mathbb{N}$ (Freiheitsgrade, Formparameter)

Merkmalraum: $M_X = \mathbb{R}^+$

Dichte:

$$f(x) = \frac{\Gamma\big((m+n)/2\big)\, m^{m/2} n^{n/2} x^{m/2-1}}{\Gamma(m/2)\Gamma(n/2)(mx+n)^{(m+n)/2}}, \quad x > 0$$

Mittelwert:

$$\mathbb{E}(X) = \frac{n}{n-2}, \quad n > 2$$

Varianz:

$$\mathrm{Var}(X) = \frac{2n^2(m+n-2)}{m(n-2)^2(n-4)}, \quad n > 4$$

Momente:

$$\mathbb{E}(X^k) = \left(\frac{n}{m}\right)^k \frac{\Gamma(m/2+k)\Gamma(n/2-k)}{\Gamma(m/2)\Gamma(n/2)}, \quad k < \frac{n}{2}$$

Quantile:

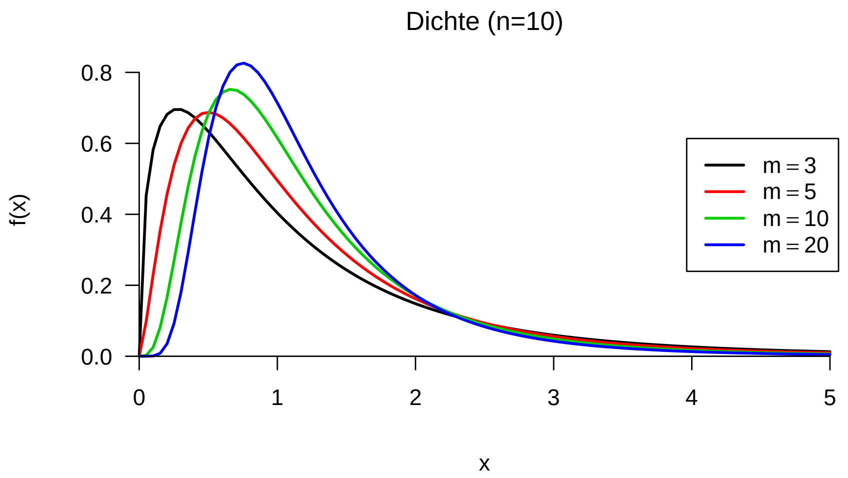
$$F_{m,n;\,p} = \frac{1}{F_{n,m;\,1-p}}, \quad 0 < p < 1$$

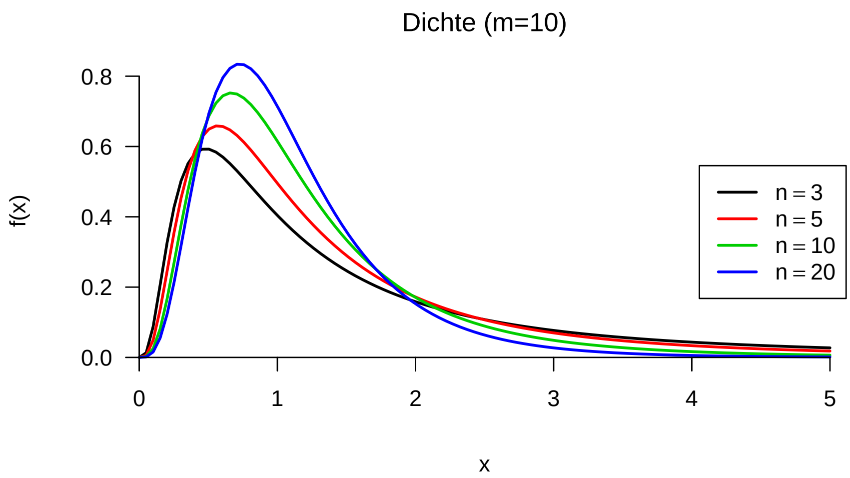
Modus:

$$x_{\mathrm{mod}} = \frac{n(m-2)}{m(n+2)}, \quad m \geq 2$$

R–Funktionen: `df1` $\widehat{=}\ m$, `df2` $\widehat{=}\ n$

```
df(x, df1, df2, log = FALSE)
pf(q, df1, df2, ncp = 0, lower.tail = TRUE, log.p = FALSE)
qf(p, df1, df2, ncp = 0, lower.tail = TRUE, log.p = FALSE)
rf(n, df1, df2, ncp = 0)
```


Dichte (n=10)
f(x)
0.0
0.2
0.4
0.6
0.8
0
1
2
3
4
5
x
m = 3
m = 5
m = 10
m = 20


Dichte (m=10)
f(x)
0.0
0.2
0.4
0.6
0.8
0
1
2
3
4
5
x
n = 3
n = 5
n = 10
n = 20

A.2.9 Betaverteilung

Bezeichnung: $X \sim Be(a, b)$, $\beta(a, b)$

Parameter: $a, b > 0$ (Formparameter)

Merkmalraum: $M_X = (0, 1)$

Betafunktion:

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \quad x, y > 0$$

$$B(x, y) = B(y, x), \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Dichte:

$$f(x) = \frac{1}{B(a, b)}\, x^{a-1}(1-x)^{b-1} I_{(0,1)}(x)$$

$$f(x|a, b) = f(1-x|b, a)$$

Verteilungsfunktion:

$$F(x) = \frac{1}{B(a, b)} \int_0^x t^{a-1}(1-t)^{b-1}\, dt, \quad x \in (0, 1) \qquad \text{[unvollständige Betafunktion]}$$

$$F(x|a, b) = 1 - F(1-x|b, a)$$

Mittelwert:

$$\mathbb{E}(X) = \frac{a}{a+b}$$

Varianz:

$$\mathrm{Var}(X) = \frac{ab}{(a+b+1)(a+b)^2}$$

Momente:

$$\mathbb{E}(X^k) = \frac{B(a+k, b)}{B(a, b)}, \quad k \in \mathbb{N}$$

Quantile:

$$Be(a, b;\, p) = 1 - Be(b, a;\, 1-p), \quad 0 < p < 1$$

Modus:

$$x_{\text{mod}} = \frac{a-1}{a+b-2}, \quad a, b > 1$$

Spezialfall: $Be(1,1) \equiv U_{0,1}$

R–Funktionen: `shape1` $\widehat{=}$ a, `shape2` $\widehat{=}$ b

```
beta(a, b)  # Betafunktion
dbeta(x, shape1, shape2, ncp = 0, log = FALSE)
pbeta(q, shape1, shape2, ncp = 0, lower.tail = TRUE, log.p = FALSE)
qbeta(p, shape1, shape2, ncp = 0, lower.tail = TRUE, log.p = FALSE)
rbeta(n, shape1, shape2, ncp = 0)
```

Dichte